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Pointwise Convergence of Multiple Fourier Series: Sufficient Conditions and an Application to Numerical Integration

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Sufficient conditions are established for the pointwise convergence of square partial sums of multiple Fourier series, in any number of dimensions, for functions with discontinuities. The result is used to extend the theory of lattice rules for multiple integration.

1. INTRODUCTION

The aim of this paper is to obtain a result on the pointwise convergence of multiple Fourier series in the presence of discontinuties of a simple kind. This result was motivated by a problem in numerical integration, to be explained in the final section. The result will be used there (see in particular Theorem 6.4) to extend the theory of so-called "lattice rules" for multiple integration.

Let $L_p^1(\mathbb{R}^s)$ denote the space of measurable functions on $\mathbb{R}^s$ that are 1-periodic in each variable, and for which

$$\int |f(x)|\, dx = \int_0^1 \cdots \int_0^1 |f(x_1, ..., x_s)|\, dx_1 \cdots dx_s < \infty.$$

(Unless stated otherwise, all integrals will be over the closed unit cube $H = \{x \in \mathbb{R}^s : 0 \leqslant x_j \leqslant 1,\ 1 \leqslant j \leqslant s\}$.) We shall assume that the elements of $L_p^1(\mathbb{R}^s)$ have well defined point values $f(x)$ for all $x \in \mathbb{R}^s$.

Given $f \in L_p^1(\mathbb{R}^s)$, the Fourier series of f is

$$f \sim \sum_{m \in \mathbb{Z}^s} \hat{f}(m)\, e^{2\pi i m . x}, \tag{1.1}$$

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where $m=(m_1, ..., m_s)$, $mx=m_1x_1+\cdots+m_sx_s$, and

$$\hat{f}(m)=\int f(x)\,e^{-2\pi imx}\,dx.$$

The Nth square partial Fourier sum is

$$S_Nf(x)=\sum_{|m|\leqslant N}\hat{f}(m)\,e^{2\pi imx},$$

where $N\in\mathbb{Z}^+$, and $|m|\leqslant N$ signifies $|m_1|, ..., |m_s|\leqslant N$. It is readily seen that

$$S_Nf(x)=\int f(y)\,D_N(x_1-y_1)\cdots D_N(x_s-y_s)\,dy,$$

where

$$D_N(t)=\sum_{k=-N}^{N}e^{2\pi ikt}=\frac{\sin\pi(2N+1)\,t}{\sin\pi t},$$

the Nth Dirichlet kernel.

Our concern is with the limit of $S_Nf(x)$ as $N\to\infty$, at a point x at which f may be discontinuous. In 1937 Cesari [2] and Tonelli [9] established, for $s=2$, conditions under which $S_Nf(x)$ converges to the mean of the four limiting values of f as x is approached from each of the open quadrants with edges parallel to the coordinate axes, assuming those limits exist. That is, they established conditions under which

$$\begin{aligned}S_Nf(x_1,x_2)&\\ \to\tfrac{1}{4}\,[f(x_1&+0,x_2+0)+f(x_1+0,x_2-0)\\ &+f(x_1-0,x_2+0)+f(x_1-0,x_2-0)].\end{aligned}\tag{1.2}$$

These conditions were expressed in terms of integrals of one-dimensional variation. These results were generalised to $s=3$ by Cesari [3], and then to all $s>2$ by Chen [4]. Here we seek a condition for convergence, again valid for all s, that is easier to apply.

For notational convenience, we consider only convergence at $x=0$. Then with f^* defined by

$$f^*(x_1,...,x_s)=\sum f(\pm x_1,...,\pm x_s),\tag{1.3}$$

the sum being taken over the 2^s possible sign combinations, our main result, Theorem 4.2, asserts that

$$S_Nf(0)\to 2^{-s}\lim_{\varepsilon\to 0}f^*(\varepsilon,...,\varepsilon)\qquad\text{as } N\to\infty,\tag{1.4}$$

provided that f is suitably well behaved near the boundary of H. A sufficient condition, employing a notion of one-sided smoothness at the boundary, is introduced in Section 5. The one-dimensional version of Theorem 4.2 is the familiar result that

$$S_N f(0) \to \tfrac{1}{2} \lim_{\varepsilon \to 0} [f(\varepsilon) + f(-\varepsilon)] \qquad \text{as } N \to \infty$$

if f has right-hand and left-hand limits at 0 and if Dini's test (see below) holds at 0.

1.1. *Localization.* One major complication with multiple Fourier series when compared with the one-dimensional case is the lack of a localization principle. Roughly speaking, localization for one-dimensional Fourier series means that the behavior of such a series at a point (or set) depends only on the values of the function in a neighborhood of that point (or set). For example, Dini's test (described below) for convergence at 0 of the Fourier series of an integrable f only requires hypotheses in an arbitrarily small neighborhood of 0.

Nothing like this is possible for $s \geqslant 2$. For example, Igari [5] showed that there exists a continuous function in $L^1_p(\mathbb{R}^s)$, $s \geqslant 2$, which is zero in a neighborhood of 0 and yet is such that

$$\limsup_N S_N f(0) = \infty.$$

Further refinements of this result are contained in Price and Shepp [6]. The surveys by Ash [1] and Zhizhiashvili [10] and the references given there contain further information on multiple Fourier series and the localization property.

The lack of a localization principle makes it inevitable that our conditions on $f \in L^1_p(\mathbb{R}^s)$ for $s > 1$ are not merely conditions on f in a neighborhood of the origin. Rather, they are conditions in a neighborhood of the boundary of H.

1.2. *Dini's Test.* Dini's test for one-dimensional Fourier series (Zygmund [11, Chap. II, (6.1)]) asserts that $S_N f(0) \to b$ if

$$\int_0^\delta \frac{|f(x) + f(-x) - 2b|}{x}\, dx < \infty \tag{1.5}$$

for some $\delta > 0$. Our analysis for the case $s > 1$ will be based on a generalisation of Dini's test, Theorem 3.1 below.

To motivate the definitions that follow, it is convenient to rewrite Dini's test as: if there exists $g \in L^1_p(\mathbb{R})$ such that $g = f$ on $(0, 1)$ and such that

$$\int_0^\delta \frac{|g(x) + g(-x) - 2g(0)|}{x}\, dx < \infty \tag{1.6}$$

for some $\delta > 0$, then $S_N f(0) \to g(0)$.

2. Preliminary Definitions

2.1. *Auxiliary Functions.* The following definitions describe multidimensional generalisations of the integrand in (1.6). Given $g \in L_p^1(\mathbb{R}^s)$, we define g^* by (1.3), and then define functions $g_j, g_{jk}, ..., g_{1 \cdots s}$ of $1, 2, ..., s$ variables respectively by

$$g_j(x_j) = \frac{g^*(0, ..., 0, x_j, 0, ..., 0) - g^*(0)}{x_j}, \qquad \text{for } 1 \leqslant j \leqslant s,$$

$$g_{jk}(x_j, x_k) = \frac{\begin{array}{c} g^*(..., 0, x_j, 0, ..., 0, x_k, 0, ...) - g^*(..., 0, x_j, 0, ..., 0, 0, 0, ...) \\ - g^*(..., 0, 0, 0, ..., 0, x_k, 0, ...) + g^*(0) \end{array}}{x_j x_k}$$

for $1 \leqslant j < k \leqslant s$, and so on, until

$$g_{1 \cdots s}(x) = \frac{\begin{array}{c} g^*(x) - g^*(x_1, ..., x_{s-1}, 0) - \cdots \\ + g^*(x_1, ..., x_{s-2}, 0, 0) + \cdots + (-1)^s g^*(0) \end{array}}{x_1 \cdots x_s}.$$

(We are not concerned with the values of these functions when the denominators are zero.) These functions are divided-difference analogues of mixed first derivatives of g^* at 0. We shall need the following identity, which is related to Taylor's theorem taken to first order in each variable:

$$g^*(x) - g^*(0) = \sum_j x_j g_j(x_j) + \sum_{j<k} x_j x_k g_{jk}(x_j, x_k) + \cdots + x_1 x_2 \cdots x_s g_{1 \cdots s}(x). \tag{2.1}$$

2.2. *The Set $\mathscr{D}$.* We shall subsequently establish convergence of $S_N f(0)$ for all f in a certain set $\mathscr{D}$, which is a multidimensional analogue of the class of functions satisfying Dini's test.

Let I denote the interval $(0, 1/2)$ and H° the interior of the unit cube H. Then $\mathscr{D}$ is the set of functions $f \in L_p^1(\mathbb{R}^s)$ for which there exists $g \in L_p^1(\mathbb{R}^s)$ such that $f = g$ on H°, and

$$\begin{array}{ll} g_j \in L^1(I) & 1 \leqslant j \leqslant s, \\ g_{jk} \in L^1(I^2) & \text{for } 1 \leqslant j < k \leqslant s, \\ \cdots\cdots\cdots & \\ g_{1 \cdots s} \in L^1(I^s), & \end{array}$$

where $g_j, g_{jk}, ...,$ are as defined above.

2.3. *Remark.* In view of the periodicity, membership of $\mathscr{D}$ imposes conditions on f in a neighborhood of the boundary of H. We shall see later (in Section 5) some reasonably natural conditions which ensure membership of $\mathscr{D}$. The following theorem, the key result of the paper, is a direct analogue of Dini's test.

3. A Dini-Like Convergence Theorem

3.1. Theorem. *Let $f \in \mathscr{D}$. Then*

$$S_N f(0) \to g(0) \qquad \text{as } N \to \infty,$$

where g is as in the definition of $\mathscr{D}$.

Proof. Since each D_N is even and all the functions are 1-periodic in each variable,

$$\begin{aligned} S_N f(0) - g(0) &= \int (f(x) - g(0))\, D_N(x_1) \cdots D_N(x_s)\, dx \\ &= \int_I \cdots \int_I (f^*(x) - g^*(0))\, D_N(x_1) \cdots D_N(x_s)\, dx. \end{aligned} \tag{3.1}$$

Now for $x \in (0, 1/2]^s$ we have, from (2.1),

$$\begin{aligned} f^*(x) - g^*(0) &= g^*(x) - g^*(0) \\ &= \sum_j x_j g_j(x_j) + \sum_{j<k} x_j x_k g_{jk}(x_j, x_k) + \cdots \\ &\quad + x_1 x_2 \cdots x_s g_{1 \ldots s}(x). \end{aligned}$$

After splitting the integral (3.1) into the various pieces according to this decomposition, we see by the generalised Riemann–Lebesgue lemma (Zygmund [11, Chap. XVII]) that each of the separate integrals converges to zero. For example,

$$\begin{aligned} &\int_I \cdots \int_I x_1 x_2 g_{12}(x_1, x_2)\, D_N(x_1) \cdots D_N(x_s)\, dx \\ &= 2^{-s+2} \int_I \int_I g_{12}(x_1, x_2) \frac{x_1}{\sin \pi x_1} \frac{x_2}{\sin \pi x_2} \sin(\pi(2N+1)\, x_1) \\ &\quad \times \sin(\pi(2N+1)\, x_2)\, dx_1\, dx_2, \end{aligned}$$

which converges to zero because $g_{12} \in L^1(I^2)$. ∎

4. One-Sided Continuity of f

In this section we give a sufficient condition for the identity

$$g(0) = 2^{-s} \lim_{\varepsilon \to 0} f^*(\varepsilon, ..., \varepsilon), \tag{4.1}$$

for $f \in \mathscr{D}$. In essence the condition is that f be one-sidedly continuous in a neighborhood of 0. When coupled with Theorem 3.1 above this allows us to state, as Theorem 4.2, a condition under which $S_N f(0)$ converges to the right-hand side of (4.1). This is the main result of this section.

4.1. Theorem. *Suppose that $f \in \mathscr{D}$, and that, for some open ball B centered at 0, f^* is continuous on $H^\circ \cap B$ and has a continuous extension to $\partial H \cap \bar{B}$. If g is as in the definition of $\mathscr{D}$, then* (4.1) *is valid.*

Proof. We denote by F^* the continuous function on $\partial H \cap \bar{B}$ defined by

$$F^*(x) = \lim_{\substack{x' \to x \\ x' \in H^\circ \cap B}} f^*(x'), \qquad x \in \partial H \cap \bar{B}. \tag{4.2}$$

For simplicity we restrict ourselves to $s = 3$, and write $(x_1, x_2, x_3) = (x, y, z)$. Let δ satisfying $0 < \delta \leqslant 1/2$ be such that $(0, \delta)^3 \subset B$. Throughout the proof $E_1, E_2, ...$ denote measurable sets in $J = (0, \delta)$, each with measure δ. Because $f \in \mathscr{D}$, we have $g_{13} \in L^1(J^2)$, and so there exists E_1 for which

$$x \mapsto z g_{13}(x, z) = \frac{g^*(x, 0, z) - g^*(0, 0, z)}{x} - g_1(x) \tag{4.3}$$

is integrable on J for $z \in E_1$. Since $g_1 \in L^1(J)$ we have

$$x \mapsto \frac{g^*(x, 0, z) - g^*(0, 0, z)}{x} \in L^1(J) \qquad \text{for } z \in E_1. \tag{4.4}$$

A similar argument starting with g_{123} shows the existence of E_2 for which

$$\begin{aligned}(x, y) &\mapsto z g_{123}(x, y, z) \\ &= \frac{g^*(x, y, z) - g^*(x, 0, z) - g^*(0, y, z) + g^*(0, 0, z)}{xy} - g_{12}(x, y)\end{aligned}$$

is integrable on J^2 for $z \in E_2$. Since g_{12} is integrable on J^2, so is the first quotient. Thus there exists E_3 for which

$$x \mapsto \frac{g^*(x, y, z) - g^*(x, 0, z) - g^*(0, y, z) + g^*(0, 0, z)}{x}$$

is integrable on J for $y \in E_3$, $z \in E_2$. Coupling this with (4.4) leads to

$$x \mapsto \frac{g^*(x, y, z) - g^*(0, y, z)}{x} \in L^1(J) \tag{4.5}$$

for $y \in E_3$, $z \in E_4 = E_1 \cap E_2$.

Now $g^*(x, y, z) = f^*(x, y, z)$ for $(x, y, z) \in H^\circ$, thus g^* is continuous on $H^\circ \cap B$ and has a continuous extension to $\partial H \cap \bar{B}$. In particular, $g^*(x, y, z)$ with $y \in E_3$ and $z \in E_4$ must have a limit as $x \to 0$. From (4.5) that limit must be $g^*(0, y, z)$, since otherwise a contradiction is easily obtained. Thus we have, using also (4.2),

$$\begin{aligned} g^*(0, y, z) &= \lim_{x \to 0} g^*(x, y, z) = \lim_{x \to 0} f^*(x, y, z) \\ &= F^*(0, y, z) \qquad \text{for } y \in E_3, z \in E_4. \end{aligned} \tag{4.6}$$

Next we show that the analogous property extends to the edge $x = y = 0$, $z \in J$, and thence to the origin. Since $g_{23} \in L^1(J^2)$, there exists E_5 such that

$$y \mapsto z g_{23}(y, z) = \frac{g^*(0, y, z) - g^*(0, 0, z)}{y} - g_2(y) \in L^1(J) \qquad \text{for } z \in E_5,$$

and hence

$$y \mapsto \frac{g^*(0, y, z) - g^*(0, 0, z)}{y} \in L^1(J) \qquad \text{for } z \in E_5. \tag{4.7}$$

For $z \in E_4$ we see from (4.6) that $g^*(0, y, z)$ is equal almost everywhere to a continuous function of y, namely $F^*(0, y, z)$. Thus (4.6) and (4.7) give

$$\begin{aligned} g^*(0, 0, z) &= \lim_{y \to 0} F^*(0, y, z) = F^*(0, 0, z) \\ &\text{for } z \in E_6 = E_4 \cap E_5. \end{aligned} \tag{4.8}$$

Finally, since

$$z \mapsto g_3(z) = \frac{g^*(0, 0, z) - g^*(0)}{z} \in L^1(J),$$

(4.8) gives

$$g^*(0) = \lim_{z \to 0} F^*(0, 0, z) = F^*(0),$$

which implies (4.1). ∎

From this theorem and Theorem 3.1 we obtain our main result:

4.2. THEOREM. *Suppose that $f \in \mathscr{D}$, and that, for some open ball B centered at 0, f^* is continuous on $H^\circ \cap B$ and has a continuous extension to $\partial H \cap \bar{B}$. Then*

$$S_N f(0) \to 2^{-s} \lim_{\varepsilon \to 0} f^*(\varepsilon, ..., \varepsilon) \qquad \text{as } N \to \infty.$$

5. ONE-SIDED SMOOTHNESS

Our aim in this section is to give sufficient conditions for membership of $\mathscr{D}$ in terms of one-sided partial derivatives of f at the boundary ∂H. Once this is done, convergence of $S_N f(0)$ follows from Theorem 4.2. The main result is Theorem 5.3. Again for notational simplicity we restrict ourselves to $s = 3$, the higher-dimensional analogues being obvious.

5.1. DEFINITION. Let $f \in L_p^1(\mathbb{R}^s)$; f is said to have *one-sided smoothness at the boundary ∂H of H* if

(i) there exists an open set V with $\bar{V} \subset H^\circ$ such that f is continuous on $H^\circ \backslash V$;

(ii) the restriction of f to $H^\circ \backslash V$ has a continuous extension to ∂H (we denote this new function on H by $\bar{f}$);

(iii) (a) at least one of the six mixed partial derivatives $\partial^3 \bar{f}/\partial x\, \partial y\, \partial z$, $\partial^3 \bar{f}/\partial y\, \partial x\, \partial z$, etc., exists in some region $U_3 = \{(x, y, z) : 0 \leqslant x, y, z < k\}$, where $k > 0$, and is continuous at $(0, 0, 0)$; and similarly for the seven remaining corner points of H;

(iii) (b) for each $0 \leqslant z \leqslant 1$ at least one of the mixed partial derivatives $\partial^2 \bar{f}/\partial x\, \partial y$, $\partial^2 \bar{f}/\partial y\, \partial x$ exists in some region $U_2 = \{(x, y, z) : 0 \leqslant x, y < k\}$, where $k = k(z) > 0$, and is continuous at $(0, 0, z)$; and similarly for the analogous partial derivatives along the remaining eleven edges of H; and

(iii) (c) for each $0 \leqslant y, z \leqslant 1$, the partial derivative $\partial \bar{f}/\partial x$ exists in some region $U_1 = \{(x, y, z) : 0 \leqslant x < k\}$, where $k = k(y, z) > 0$, and is continuous at $(0, y, z)$; and similarly for the partial derivatives on the remaining five faces of H.

A word of explanation is in order here. When considering the existence of the above partial derivatives on the boundary of the various regions, we ask only for the corresponding one-sided limits. For example, a requirement of (iii) (b) is that for each $0 \leqslant z \leqslant 1$ (and assuming we are dealing with $\partial^2 \bar{f}/\partial x\, \partial y$), the double limit

$$\lim_s \lim_t \frac{\bar{f}(x+s, y+t, z) - \bar{f}(x+s, y, z) - \bar{f}(x, y+t, z) + \bar{f}(x, y, z)}{st}$$

exists as $s, t \to 0$ when $0 < x, y < k$; as $s \to 0$, $t \to 0+$ when $0 < x < k$, $y = 0$; as $s \to 0+$, $t \to 0$ when $x = 0$, $0 < y < k$; and as $s, t \to 0+$ when $x, y = 0$. Denoting this limit by $\partial^2 \bar{f}/\partial x \, \partial y$ in all cases, a further requirement is that

$$\lim_{x, y \to 0+} \frac{\partial^2 \bar{f}}{\partial x \, \partial y}(x, y, z) = \frac{\partial^2 \bar{f}}{\partial x \, \partial y}(0, 0, z).$$

Finally, (iii) (b) asks that similar conditions hold at each of the other edges of H.

Before stating the next lemma, it is convenient to introduce the following notation corresponding to a function f with one-sided smoothness at the boundary ∂H. For each $y, z \in [0, 1]$, define $A_{1,y,z}$ on $[0, 1]$ by

$$A_{1,y,z}(x) = \bar{f}(x, y, z) - \bar{f}(0, y, z);$$

for each $z \in [0, 1]$, define $A_{2,z}$ on $[0, 1] \times [0, 1]$ by

$$A_{2,z}(x, y) = \bar{f}(x, y, z) - \bar{f}(0, y, z) - \bar{f}(x, 0, z) + \bar{f}(0, 0, z);$$

and finally define A_3 on $[0, 1] \times [0, 1] \times [0, 1]$ by

$$\begin{aligned} A_3(x, y, z) = {} & \bar{f}(x, y, z) - \bar{f}(0, y, z) - \bar{f}(x, 0, z) - \bar{f}(x, y, 0) \\ & + \bar{f}(0, 0, z) + \bar{f}(0, y, 0) + \bar{f}(x, 0, 0) - \bar{f}(0, 0, 0). \end{aligned}$$

We also define the analogous functions on the remaining faces, edges, and corners of H.

5.2. Lemma. *Let $f \in L_p^1(\mathbb{R}^3)$ have one-sided smoothness ∂H. Then*

$$\lim_{x \to 0+} A_{1,y,z}(x)/x$$

exists for each $0 \leqslant y$, $z \leqslant 1$, and similarly for the remaining faces;

$$\lim_{x, y \to 0+} A_{2,z}(x, y)/xy$$

exists for each $0 \leqslant z \leqslant 1$, and similarly for the remaining edges; and

$$\lim_{x, y, z \to 0+} A_3(x, y, z)/xyz$$

exists, and similarly for the remaining corners.

Proof. For notational convenience, we denote $\bar{f}$ by f. We start with the proof of the existence of the third limit.

Assume that $\partial^3 f/\partial z \, \partial y \, \partial x$ is the partial derivative referred to in condition (iii) (a). Given $0 \leqslant b$, $c < k$, define α on $[0, k)$ by

$$\alpha(a) = f(a, b, c) - f(a, b, 0) - f(a, 0, c) + f(a, 0, 0).$$

Then $A_3(a, b, c) = \alpha(a) - \alpha(0)$. Fix $a, b, c \in (0, k)$. Now α is differentiable on $(0, a)$ and continuous on $[0, a]$, so that by the mean value theorem (MVT) there exists $a_0 = a_0(a, b, c) \in (0, a)$ with

$$A_3(a, b, c) = a\alpha'(a_0).$$

Since

$$\alpha'(a_0) = (\partial/\partial x)(f(a_0, b, c) - f(a_0, b, 0)) - (\partial/\partial x)(f(a_0, 0, c) - f(a_0, 0, 0)),$$

another application of the MVT gives the existence of $b_0 \in (0, b)$ with

$$A_3(a, b, c) = ab(\partial^2/\partial y\, \partial x)(f(a_0, b_0, c) - f(a_0, b_0, 0)).$$

A final application of the same theorem gives the existence of $c_0 \in (0, c)$ with

$$A_3(a, b, c) = abc \frac{\partial^3 f}{\partial z\, \partial y\, \partial x}(a_0, b_0, c_0).$$

Hence for each point $(a, b, c) \in (0, k)^3$ we have proved the existence of (a_0, b_0, c_0) with $0 < a_0 < a$, $0 < b_0 < b$, $0 < c_0 < c$ such that

$$\frac{A_3(a, b, c)}{abc} = \frac{\partial^3 f}{\partial z\, \partial y\, \partial x}(a_0, b_0, c_0).$$

By virtue of the continuity of this final partial derivative at $(0, 0, 0)$, we obtain the required limit.

Now suppose that $\partial^2 f/\partial y\, \partial x$ is the partial derivative referred to in 5.1 (iii) (b). The analogous conclusion is that for all $z \in [0, 1]$ and $a, b \in (0, k)$ there exist a_0, b_0 with $0 < a_0 < a$, $0 < b_0 < b$ such that

$$\frac{A_{2,z}(a, b)}{ab} = \frac{\partial^2 f}{\partial y\, \partial x}(a_0, b_0, z). \tag{5.1}$$

The required limit follows from the continuity of the partial derivative at $(0, 0, z)$.

Finally, from 5.1 (iii) (c) the MVT shows that for all $y, z \in [0, 1]$ and $a \in (0, k)$ there exists $a_0 \in (0, a)$ such that

$$\frac{A_{1,y,z}(a)}{a} = \frac{\partial f}{\partial x}(a_0, y, z), \tag{5.2}$$

and the required conclusion follows. ∎

The main result of this section, Theorem 5.3 below, requires a slightly stronger property than one-sided smoothness. Once more we make the restriction that $s=3$. A function $f \in L_p^1(\mathbb{R}^s)$ is said to have *uniform* one-sided smoothness at the boundary ∂H of H if it satisfies the following condition in addition to the conditions of Definition 5.1:

(iv) In condition (iii) (b), $\min\{k(z)\colon 0 \leqslant z \leqslant 1\} > 0$, and the derivative $\partial^2 \tilde{f}/\partial x\, \partial y$ (or $\partial^2 \tilde{f}/\partial y\, \partial x$) satisfies the property that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{\partial^2 \tilde{f}}{\partial x\, \partial y}(x, y, z) - \frac{\partial^2 \tilde{f}}{\partial x\, \partial y}(0, 0, z)\right| < \varepsilon$$

for $0 \leqslant z \leqslant 1$ and $0 \leqslant x, y \leqslant \delta$, and similarly for the remaining edges. Also, in condition (iii) (c), $\min\{k(y, z)\colon 0 \leqslant y, z \leqslant 1\} > 0$, and $\partial \tilde{f}/\partial x$ has the property that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{\partial \tilde{f}}{\partial x}(x, y, z) - \frac{\partial \tilde{f}}{\partial x}(0, y, z)\right| < \varepsilon$$

for $0 \leqslant y$, $z \leqslant 1$ and $0 \leqslant x \leqslant \delta$, and similarly for the remaining faces.

Under the assumption of uniform one-sided smoothness, the identities (5.1) and (5.2) can be used to strengthen the first two conclusions of Lemma 5.2. They become: for each $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\left|\frac{A_{1,y,z}(x)}{x} - \frac{\partial \tilde{f}}{\partial x}(0, y, z)\right| < \varepsilon \tag{5.3}$$

for y, $z \in [0, 1]$ and $0 < x < \delta$ (and similarly for the remaining faces); and for each $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\left|\frac{A_{2,z}(x, y)}{xy} - \frac{\partial^2 \tilde{f}}{\partial y\, \partial x}(0, 0, z)\right| < \varepsilon \tag{5.4}$$

for $z \in [0, 1]$ and $0 < x, y < \delta$ (and similarly for the remaining edges).

5.3. THEOREM. *Assume that $f \in L_p^1(\mathbb{R}^s)$ has uniform one-sided smoothness at the boundary of H. Then $f \in \mathscr{D}$, and*

$$S_N f(0) \to 2^{-s} \lim_{\varepsilon \to 0} f^*(\varepsilon, ..., \varepsilon) \qquad \text{as } N \to \infty.$$

Proof. All that is required is to show that $f \in \mathscr{D}$, since Theorem 4.2 can then be applied. As for the present we are only working within the cube H, we denote $\tilde{f}$ by f.

First notice that there exist δ_1, $K_1 > 0$ such that

$$|f(x, y, z) - f(0, y, z)| \leqslant K_1 x \qquad \text{for } 0 \leqslant x \leqslant \delta_1,\ 0 \leqslant y,\ z \leqslant 1. \tag{5.5}$$

(This follows directly from (5.3) since $\partial f/\partial x$ is continuous, and hence bounded, on $\{(0, y, z)\colon 0 \leqslant y,\ z \leqslant 1\}$.) Similarly there exist δ_2, K_2, δ_3, $K_3 > 0$ such that

$$|f(x, y, z) - f(x, 0, z)| \leqslant K_2 y \qquad 0 \leqslant y \leqslant \delta_2, \qquad 0 \leqslant x,\ z \leqslant 1, \tag{5.6}$$

$$|f(x, y, z) - f(x, y, 0)| \leqslant K_3 z \qquad \text{for } 0 \leqslant z \leqslant \delta_3, \quad 0 \leqslant x,\ y \leqslant 1. \tag{5.7}$$

We now assert the existence of δ_4, $K_4 > 0$ such that

$$|f(x, y, z) - f(x, 0, z) - f(0, y, z) + f(0, 0, z)| \leqslant K_4 xy$$
$$\text{for } 0 \leqslant x,\ y \leqslant \delta_4,\ 0 \leqslant z \leqslant 1. \tag{5.8}$$

(Just as (5.5) follows from (5.3), so the stated inequality follows from (5.4).) Similarly, there exist δ_5, K_5, δ_6, $K_6 > 0$ such that

$$|f(x, y, z) - f(x, y, 0) - f(0, y, z) + f(0, y, 0)| \leqslant K_5 xz$$
$$\text{for } 0 \leqslant x,\ z \leqslant \delta_5,\ 0 \leqslant y \leqslant 1, \tag{5.9}$$

$$|f(x, y, z) - f(x, y, 0) - f(x, 0, z) + f(x, 0, 0)| \leqslant K_6 yz$$
$$\text{for } 0 \leqslant y,\ z \leqslant \delta_6,\ 0 \leqslant x \leqslant 1. \tag{5.10}$$

Finally, use of the existence of the third limit in Lemma 5.2 establishes the existence of δ_7, $K_7 > 0$ such that

$$|f(x, y, z) - f(x, y, 0) - f(x, 0, z) - f(0, y, z) + f(x, 0, 0) + f(0, y, 0)$$
$$+ f(0, 0, z) - f(0, 0, 0)| \leqslant K_7 xyz \qquad \text{for } 0 \leqslant x,\ y,\ z \leqslant \delta_7. \tag{5.11}$$

Let $\delta = \min(\delta_1, ..., \delta_7)$. Denote by J the integral

$$J = \int_H \frac{|f(x, y, z) - f(x, y, 0) - \cdots + f(x, 0, 0) + \cdots - f(0, 0, 0)|}{xyz}\, dx\, dy\, dz.$$

Write J as $J = J_1 + J_2 + \cdots + J_8$, where each J_j is formed by restricting the integrand to the regions

$J_1 : 0 \leqslant x, y, z \leqslant \delta$,

$J_2 : 0 \leqslant x, y \leqslant \delta < z \leqslant 1$, and similarly for J_3, J_4,

$J_5 : 0 \leqslant x \leqslant \delta < y, z \leqslant 1$, and similarly for J_6, J_7,

$J_8 : \delta < x, y, z \leqslant 1$.

For each j, denote the region corresponding to J_j by H_j. Using (5.11), $J_1 < \infty$. For J_2, apply (5.8) to get

$$J_2 \leqslant \frac{1}{\delta} \int_{H_2} \frac{|f(x, y, z) - f(x, 0, z) - f(0, y, z) + f(0, 0, z)|}{xy}\, dx\, dy\, dz$$

$$+ \frac{1}{\delta} \int_{H_2} \frac{|f(x, y, 0) - f(x, 0, 0) - f(0, y, 0) + f(0, 0, 0)|}{xy}\, dx\, dy\, dz$$

$$\leqslant \frac{2}{\delta} \delta^2 (1 - \delta)\, K_4 < \infty.$$

Similarly, $J_3, J_4 < \infty$ by the use of (5.9) and (5.10).

Turning to J_5, application of (5.5) gives

$$J_5 \leqslant \frac{1}{\delta^2} \int_{H_5} \frac{|f(x, y, z) - f(0, y, z)|}{x}\, dx\, dy\, dz$$

$$+ \frac{1}{\delta^2} \int_{H_5} \frac{|f(x, y, 0) - f(0, y, 0)|}{x}\, dx\, dy\, dz$$

$$+ \frac{1}{\delta^2} \int_{H_5} \frac{|f(x, 0, z) - f(0, 0, z)|}{x}\, dx\, dy\, dz$$

$$+ \frac{1}{\delta^2} \int_{H_5} \frac{|f(x, 0, 0) - f(0, 0, 0)|}{x}\, dx\, dy\, dz$$

$$\leqslant \frac{4}{\delta^2} \delta (1 - \delta)^2 K_1 < \infty.$$

Similarly, from (5.6) and (5.7), $J_6, J_7 < \infty$. Finally, $J_8 < \infty$ since f is integrable on H and $(xyz)^{-1}$ is bounded on H_8. Hence

$$J < \infty. \tag{5.12}$$

The last step is to define $g \in L_p^1(\mathbb{R}^3)$ by $g = f$ on H°, and by a suitable limiting process on ∂H. Specifically, for $(x, y, z) \in \partial H$ we set

$$g(x, y, z) = \tfrac{1}{8} \lim_{\varepsilon \to 0} \sum f(x \pm \varepsilon, y \pm \varepsilon, z \pm \varepsilon),$$

the sum being taken over the 8 possible sign combinations. For example,

$$g(0, 0, 0) = \tfrac{1}{8} [\tilde{f}(0, 0, 0) + \tilde{f}(1, 0, 0) + \cdots + \tilde{f}(1, 1, 1)],$$

$$g(x, 0, 0) = \tfrac{1}{4} [\tilde{f}(x, 0, 0) + \tilde{f}(x, 1, 0) + \tilde{f}(x, 0, 1) + \tilde{f}(x, 1, 1)],$$

$$g(x, y, 0) = \tfrac{1}{2} [\tilde{f}(x, y, 0) + \tilde{f}(x, y, 1)],$$

for $x, y \in (0, 1)$, where we now revert to the notation $\tilde{f}$ for the extension of f to the boundary of H. It is then easily seen that for all $(x, y, z) \in H$ we have

$$\begin{aligned} g^*(x, y, z) = {} & \tilde{f}(x, y, z) + \tilde{f}(1-x, y, z) + \tilde{f}(x, 1-y, z) + \tilde{f}(1-x, 1-y, z) \\ & + \tilde{f}(x, y, 1-z) + \tilde{f}(1-x, y, 1-z) + \tilde{f}(x, 1-y, 1-z) \\ & + \tilde{f}(1-x, 1-y, 1-z). \end{aligned}$$

Since g^* restricted to H is a sum of 8 functions each of which has uniform one-sided smoothness at ∂H, that $g_{123} \in L^1(I^3)$ follows from the finiteness of the above integral J and the 7 analogous integrals obtained by replacing $f(x, y, z)$ by $f(1-x, y, z)$, etc.

Similar steps show that $g_{12}, g_{13}, g_{23} \in L^1(I^2)$ and that $g_1, g_2, g_3 \in L^1(I)$. Hence $f \in \mathscr{D}$. The other conditions of Theorem 4.2 are evidently satisfied, thus the result follows. ∎

6. Lattice Rules for Multiple Integration

We consider the numerical approximation of

$$IF = \int F(x)\, dx \tag{6.1}$$

over the closed unit cube H, with $F \in C(H)$, the set of continuous functions on H, by a "lattice rule." The notion of a lattice rule, described below, was introduced in [7, 8] for functions F which have a smooth periodic extension. Theorem 6.2 and 6.4 below extend the class of admissible functions F. Before describing the approximation, we re-express the integral (6.1) in terms of a periodic function $f \in L_p^1(\mathbb{R}^s)$, defined by

$$f(x) = F(x), \qquad x \in H^\circ, \tag{6.2}$$

$$f(x) = f(x+z), \qquad x \in \mathbb{R}^s,\ z \in \mathbb{Z}^s,\ x+z \in H^\circ, \tag{6.3}$$

$$f(x) = 2^{-s} \lim_{\varepsilon \to 0} \sum f(x_1 \pm \varepsilon, ..., x_s \pm \varepsilon), \qquad x \in \partial H, \tag{6.4}$$

with the sum taken over the 2^s possible sign combinations. From the first of these follows

$$IF = If. \tag{6.5}$$

Note that the values of f are easily computed; for example, for $s = 3$ we have

$$\begin{aligned} f(0, 0, 0) = \tfrac{1}{8} [& F(0, 0, 0) + F(1, 0, 0) + F(0, 1, 0) + F(0, 0, 1) \\ & + F(1, 1, 0) + F(1, 0, 1) + F(0, 1, 1) + F(1, 1, 1)], \end{aligned}$$

$$f(x, 0, 0) = \tfrac{1}{4}\,[F(x, 0, 0) + F(x, 1, 0) + F(x, 0, 1) + F(x, 1, 1)] \quad \text{for } x \in (0, 1),$$

$$f(x, y, 0) = \tfrac{1}{2}\,[F(x, y, 0) + F(x, y, 1)] \quad \text{for } x, y \in (0, 1),$$

$$f(x, y, z) = F(x, y, z) \quad \text{for } x, y, z \in (0, 1).$$

In words, f on ∂H is the mean of the values of F at points on opposite faces.

A lattice rule for the numerical integration of the periodic function f is an equal-weight quadrature rule of the form

$$Q_L f = \frac{1}{n} \sum_{j=0}^{n-1} f(x^{(j)}), \tag{6.6}$$

where

$$\{x^{(0)}, ..., x^{(n-1)}\} = L \cap [0, 1)^s,$$

and L is an "integration lattice"; that is, L is a lattice in $\mathbb{R}^s$ (i.e., a discrete subset of $\mathbb{R}^s$ which is a group under addition) which also contains $\mathbb{Z}^s$ as a subset.

6.1. *The Case of Absolute Convergence.* For an f with an absolutely convergent Fourier series, the following result for the error in the lattice rule is already known:

THEOREM [8]. *Suppose $f \in L_p^1(\mathbb{R}^s)$ has an absolutely convergent Fourier series* (1.1). *Then*

$$Q_L f - If = \sum_{m \in L^\perp \setminus \{0\}} \hat{f}(m).$$

Here $L^\perp$ is the dual lattice of L, defined by

$$L^\perp = \{m \in \mathbb{R}^s : mx \in \mathbb{Z} \ \forall x \in L\}.$$

It is a subset of $\mathbb{Z}^s$ because $L \supset \mathbb{Z}^s$.

Since absolute convergence implies uniform convergence of the Fourier series, the absolute convergence hypothesis in Theorem 6.1 implies $f \in C(\mathbb{R}^s)$. Thus we must abandon that hypothesis if we are to handle the case of a discontinuous f introduced at the start of this section. In its place we shall impose alternative conditions which ensure pointwise convergence, in an appropriate sense, of the Fourier series of f. In the preceding sections the convergence has been studied only at 0, thus it is convenient to express the alternative conditions in terms of the translates f_x of f defined by

$$f_x(y) = f(y + x), \qquad x, y \in \mathbb{R}^s. \tag{6.7}$$

6.2. THEOREM. *Let $F \in C(H)$, and define $f \in L_p^1(\mathbb{R}^s)$ by* (6.2)–(6.4). *Suppose that for each $x \in \mathbb{R}^s$ the translate f_x defined by* (6.7) *satisfies the hypotheses of Theorem* 4.2. (*The radius of B is allowed to depend on x.*) *Then*

$$Q_L f - If = \lim_{N \to \infty} \sum_{\substack{m \in L^\perp \setminus \{0\}, \\ |m| \leqslant N}} \hat{f}(m). \tag{6.8}$$

Proof. Theorem 4.2 applied to f_x gives the conclusion, expressed in terms of f, that

$$S_N f(x) \to 2^{-s} \lim_{\varepsilon \to 0} \sum f(x_1 \pm \varepsilon, ..., x_s \pm \varepsilon) \quad \text{as } N \to \infty.$$

Since the property (6.4) holds for all $x \in \mathbb{R}^s$ (note that f is continuous in H°), we deduce

$$S_N f(x) \to f(x) \qquad \text{as } N \to \infty \ \forall x \in \mathbb{R}^s. \tag{6.9}$$

Thus we obtain

$$\begin{aligned} Q_L f &= Q_L \left(\lim_{N \to \infty} \sum_{|m| \leqslant N} \hat{f}(m)\, e^{2\pi i m x} \right) \\ &= \lim_{N \to \infty} \sum_{|m| \leqslant N} \hat{f}(m)\, Q_L(e^{2\pi i m x}). \end{aligned}$$

By Theorem 1 of [8] we have

$$Q_L(e^{2\pi i m x}) = \begin{cases} 1 & \text{if } m \in L^\perp, \\ 0 & \text{otherwise,} \end{cases}$$

thus

$$Q_L f = \lim_{N \to \infty} \sum_{\substack{m \in L^\perp \\ |m| \leqslant N}} \hat{f}(m),$$

and the result follows on observing that $\hat{f}(0) = If$. ∎

6.3. *Remark.* The key to the proof is that the pointwise convergence property (6.9) holds for all x, including points of discontinuity.

The following result is weaker but more easily understood.

6.4. THEOREM. *Suppose that $F \in C(H)$, and that all the mixed partial derivatives of the form*

$$\frac{\partial^t F}{\partial x_{i_1} \cdots \partial x_{i_t}}, \qquad 1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant s,\ 1 \leqslant t \leqslant s,$$

are continuous on H, where partial derivatives on the boundary ∂H are defined in the appropriate one-sided sense. Then with f defined by (6.2)–(6.4),

$$Q_L f - If = \lim_{N \to \infty} \sum_{\substack{m \in L^\perp \setminus \{0\},\\ |m| \leqslant N}} \hat{f}(m).$$

Proof. First note that the function $f_0 = f$ has uniform one-sided smoothness at the boundary of H (see Section 5). For general values of x the translate f_x does not have one-sided smoothness at the boundary of H, because f_x is not continuous on H: for example, it has discontinuities on the hyperplanes $y_i = -x_i$, $i = 1, ..., s$, if $-x \in H^\circ$. However, because f_x is well behaved on each of the closed rectangular solids bounded by these hyperplanes and ∂H, a modification of Theorem 5.3, in which the hypotheses are allowed to be satisfied only in an appropriate piecewise sense, can be used to establish that $f_x \in \mathscr{D}$. (We omit the details.) Since the hypotheses of Theorem 4.2 are satisfied, the result now follows from Theorem 6.2. ∎

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